Adaptive Control Class – Mini-Project 3

Modular Adaptive Safety-Critical Control (ISS approach) and Robust Control under Actuation Uncertainty

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Based on Chapters 6 and 7 of "Adaptive and Learning-Based Control of Safety-Critical Systems" by Max Cohen and Calin Belta

Abstract

This mini-project tackles two advanced paradigms for safety-critical control under uncertainties. The first part (Chapter 6) uses a modular approach built on Input-to-State Stability (ISS) concepts: an exponentially ISS Control Lyapunov Function (eISS-CLF) and an ISSf-Control Barrier Function (ISSf-CBF) are designed to cope with parameter estimation errors using various adaptive laws (gradient descent, recursive least squares with constant and variable forgetting). The second part (Chapter 7) addresses multiplicative actuation uncertainties via robust min-max optimization, converted to standard quadratic programs (QPs) through duality. Set Membership Identification (SMID) is used to tighten uncertainty bounds online. Extensive simulations and comparisons are required to understand the trade-offs between modular adaptivity and robust min-max design.

1 Project Overview

This project combines two powerful frameworks for adaptive and robust safety-critical control:

(1) **Modular Adaptive Control using ISS** (Ch. 6): The system contains parametric uncertainties. We design an eISS-CLF and ISSf-CBF that treat the parameter estimation error as an ISS input, preserving stability and safety without requiring perfect cancellation. Several parameter estimators (gradient descent, RLS, RLS with forgetting) enhanced with a history stack are compared.

(2) **Robust Control under Actuation Uncertainty** (Ch. 7): Multiplicative uncertainty in the input channels (e.g., unknown control effectiveness) is considered. Robust CLF/CBF conditions are formulated as min-max QPs, which are transformed into computationally tractable convex QPs via duality. The uncertainty bounds are refined online using Set Membership Identification (SMID) with a history stack.

The overall system is a MIMO nonlinear model with both parametric and actuation uncertainties, validated against the literature.

2 System Description

Consider a MIMO nonlinear system subject to both *parametric uncertainty* and *multiplicative actuation uncertainty*:

$$\boxed{\dot{x} = f_0(x) + F(x)\,\theta + g(x)\,(I_m + \Delta)\,u,} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, $\Delta \in \mathbb{R}^{m\times m}$ is an unknown constant (or slowly varying) matrix representing multiplicative input uncertainty, I_m is the identity, and f_0, F, g are known smooth functions. The uncertainty Δ belongs to a known convex compact set $\mathcal{D}$ (e.g., $\|\Delta\| \le \delta_{\max}$). For the modular adaptive part (Part 1), we treat $\Delta \equiv 0$ and focus on θ. In the robust part (Part 2), we assume either θ is known or an estimator provides a bounded error, and tackle Δ.

Validation: Reproduce the open-loop response from the literature and verify that both the parametric and actuation uncertainty models produce physically meaningful behavior.

3 Part 1: Modular Adaptive Safety-Critical Control via ISS

3.1 Task 1.1: eISS-CLF Design and QP

An *exponentially ISS Control Lyapunov Function* (eISS-CLF) $V(x)$ for the system (1) with $\Delta \equiv 0$ satisfies

$$\inf_{u\in\mathbb{R}^m} \left[\dot{V}_0(x,u,\hat{\theta})\right] \le -cV(x) + \iota\big(x\big), \tag{2}$$

where $\dot{V}_0(x,u,\hat{\theta}) = \omega(x) + \phi(x)^\top\hat{\theta} + L_gV(x)u$, $\tilde{\theta} = \theta - \hat{\theta}$, $c > 0$, and ι is a class $\mathcal{K}$ function. A common choice is $\iota(x) = \frac{1}{\kappa}\|\phi(x)\|^2$ for some $\kappa > 0$.

QP formulation:

$$\begin{aligned} \min_{u} \quad & \|u\|^2 \\ \text{s.t.} \quad & \omega(x) + \phi(x)^\top\hat{\theta} + L_gV(x)u \le -cV(x) + \iota\big(x\big). \end{aligned} \tag{3}$$

3.2 Task 1.2: Parameter Estimation Algorithms

You will implement and compare **four** parameter estimation algorithms, all incorporating a *history stack* of recorded data to enhance convergence.

(a) **Gradient Descent with history stack**

(b) **Recursive Least Squares (RLS) with history stack**:

(c) **RLS with Constant Forgetting Factor**:

(d) **RLS with Variable Forgetting Factor**: Implement a forgetting factor that adapts based on the prediction error (e.g., using the Fortescue algorithm or similar) to prevent covariance blow-up when excitation is low. Describe your chosen strategy.

For all algorithms, maintain a history stack of $N \geq p$ data points chosen to keep the minimum eigenvalue of $\sum \phi_j \phi_j^\top$ above a threshold. Compare the parameter convergence rate, control effort, and stability margins.

3.3 Task 1.3: ISSf-CBF and Safety Filter

Given a safety constraint $h(x) \geq 0$, its derivative under $\Delta \equiv 0$ is

$$\dot{h} = L_f h(x) + \phi_h(x)^\top \theta + L_g h(x) u, \tag{4}$$

with $\phi_h(x) = \big(L_F h(x)\big)^\top$. The *ISSf-CBF* condition demands

$$\sup_{u \in \mathbb{R}^m} \Big[L_f h(x) + \phi_h(x)^\top \hat{\theta} + L_g h(x) u \Big] \geq -\alpha_B\big(h(x)\big) + \iota_h\big(x\big). \tag{5}$$

Safety filter QP: Given a nominal (potentially unsafe) controller u_{nom} (e.g., the eISS-CLF minimizer), solve

$$\begin{aligned} \min_{u} \quad & \|u - u_{\text{nom}}\|^2 \\ \text{s.t.} \quad & L_f h(x) + \phi_h(x)^\top \hat{\theta} + L_g h(x) u \geq -\alpha_B\big(h(x)\big) + \iota_h\big(x\big). \end{aligned} \tag{6}$$

Combine with input constraints if desired. Simulate a safety-critical scenario, compare the performance of the ISSf-CBF with a naive certainty-equivalence CBF when the parameter estimate is initially poor.

3.4 Task 1.4: Comparison of Estimators

Run the closed-loop system (eISS-CLF + ISSf-CBF) with each of the four estimation algorithms. Produce the following comparison plots:

- Parameter estimation error $\|\tilde{\theta}(t)\|$.
- Evolution of the minimum eigenvalue of the history stack data matrix.
- State convergence $\|x(t)\|$ and safety margin $\min_t h(x(t))$.
- Control effort $\int_0^T \|u(t)\|^2 dt$.

Discuss which estimator provides the best trade-off between adaptation speed, excitation utilization, and robustness.

4 Part 2: Robust Safety-Critical Control under Actuation Uncertainty

In this part, the system (1) includes a multiplicative uncertainty $\Delta \in \mathcal{D} \subset \mathbb{R}^{m\times m}$. The exact value of Δ is unknown, but a bounded set is initially known. For simplicity, assume θ is known (or an accurate estimate is available from Part 1). The emphasis is on robustifying the CLF and CBF against Δ.

4.1 Task 2.1: Robust CLF via Min-Max QP

The CLF derivative now reads

$$\dot{V} = \omega(x) + \phi(x)^\top \theta + L_g V(x)(I_m + \Delta)u. \tag{7}$$

To guarantee stability for all admissible $\Delta \in \mathcal{D}$, we impose

$$\omega(x) + \phi(x)^\top \theta + \min_{\Delta\in\mathcal{D}} \big[L_g V(x)(I_m + \Delta)u\big] \leq -\gamma V(x). \tag{8}$$

This leads to a **min-max** (or min over u, max over Δ of $\dot{V}$) formulation. The robust CLF constraint becomes

$$\omega(x) + \phi(x)^\top \theta + L_g V(x)u + \min_{\Delta\in\mathcal{D}} \big[L_g V(x)\Delta u\big] \leq -\gamma V(x). \tag{9}$$

For typical sets $\mathcal{D}$ (e.g., $\|\Delta\|_2 \leq \delta_{\max}$ or $\|\Delta\|_F \leq \delta_{\max}$), the minimization can be performed analytically. For instance, if $\mathcal{D} = \{\Delta : \|\Delta\|_2 \leq \delta_{\max}\}$, then

$$\min_{\|\Delta\|_2\leq\delta_{\max}} L_g V(x)\Delta u = -\delta_{\max}\|L_g V(x)\|\|u\|. \tag{10}$$

Hence the robust CLF constraint becomes

$$\omega(x) + \phi(x)^\top \theta + L_g V(x)u - \delta_{\max}\|L_g V(x)\|\|u\| \leq -\gamma V(x). \tag{11}$$

Unfortunately, this is **non-convex** in u due to the $-\|u\|$ term. We may adopt the duality-based method from Chapter 7 and convert it to a standard QP. When $\mathcal{D}$ is described by a set of linear matrix inequalities, duality can be used to obtain an equivalent convex constraint.

Deliverable: Derive the dual QP for the case $\mathcal{D} = \{\Delta : \|\Delta\|_2 \leq \delta_{\max}\}$ (or a similar simple set), implement it, and verify that it robustly stabilizes the system for all Δ in the set.

4.2 Task 2.2: Robust CBF under Actuation Uncertainty

Analogously, for the safety constraint $h(x)$, we require

$$\min_{\Delta\in\mathcal{D}} \big[L_f h + \phi_h^\top \theta + L_g h(I + \Delta)u\big] \geq -\alpha_B(h(x)). \tag{12}$$

This again yields a robust constraint that must be converted to a convex form via duality. The robust safety filter then solves

$$\begin{aligned} \min_{u} \quad & \|u - u_{\text{nom}}\|^2 \\ \text{s.t.} \quad & \inf_{\Delta \in \mathcal{D}} \left[L_f h + \phi_h^\top \theta + L_g h (I + \Delta) u \right] \geq -\alpha_B(h(x)). \end{aligned} \tag{13}$$

Derive the dual QP, implement it, and test it in a scenario where the nominal CLF controller would violate safety under the worst-case Δ.

4.3 Task 2.3: Set Membership Identification (SMID) for Uncertainty Reduction

The initial set $\mathcal{D}$ may be conservative. To reduce conservatism online, we use **Set Membership Identification** (SMID). The idea: as the system evolves, we collect input-output data that must be consistent with the true Δ.

Implement SMID to update the bound $\delta_{\max}$ (or the set $\mathcal{D}$) online and feed it into the robust CLF and CBF QPs. Demonstrate that as the bound shrinks, the controller becomes less conservative (lower control effort, less safety margin inflation) while still robust.

5 Comprehensive Analysis and Discussion

Your report must include a thorough discussion covering:

- **Modularity vs. Robustness:** Compare the ISS-based modular approach (Part 1) with the min-max robust approach (Part 2). Under what conditions is one preferable?
- **Estimator Comparison:** Which estimation algorithm (gradient, RLS variants) is best suited for the eISS-CLF framework? How does forgetting factor influence transient behavior?
- **History Stack in Both Parts:** The history stack plays different roles (enabling CL, maintaining data for SMID). Discuss its implementation and impact.
- **Duality Derivation:** Show the key steps converting the robust CLF/CBF constraint to a QP. What are the computational trade-offs?
- **SMID Effect:** Quantify how the reduction in uncertainty bound improves performance (e.g., smaller control magnitude, closer to nominal). Show the evolution of the feasible set.
- **Practical Issues:** Noise sensitivity, discretization effects, QP feasibility, and computational load.

Table 1: Project Deliverables

Deliverable	**Format**	**Details**
Handwritten Report	PDF scan	Maximum 12 pages, single-sided, legible handwriting.
Video Presentation	MP4 file	7 minutes (strict), explaining code and key results.
MATLAB/Simulink Code	`.m` and `.slx` files	All files, archived with folder structure; executable via `Main.m`.

6 Code Organization

All simulations must run from a single `Main.m` script. Suggested file structure:

```
Main.m                         # Master script
System_Parameters.m            # True parameters, uncertainty bounds
Dynamics.m                     # f0, F, g for uncertain system
Regressors.m                   # phi(x), omega(x), LgV, etc.
eISS_CLF_Controller.m          # Task 1.1: eISS-CLF QP
Estimators/                    # Folder with estimation algorithms
   Gradient_Estimator.m
   RLS_Estimator.m
   RLS_ConstForget.m
   RLS_VariableForget.m
HistoryStack.m                 # Data storage/selection logic
ISSf_CBF_SafetyFilter.m        # Task 1.3: ISSf-CBF QP
Robust_CLF_Controller.m        # Task 2.1: robust CLF min-max QP
Robust_CBF_SafetyFilter.m      # Task 2.2: robust CBF QP
SMID.m                         # Task 2.3: set membership identification
Plot_Results.m
Simulink_Models/               # All .slx files
README.txt
```

7 Report Structure

Your handwritten report should be organized as follows:

```
1. Introduction
2. System Model and Validation
3. Part 1: Modular Adaptive Control via ISS
   3.1 eISS-CLF Design and QP
   3.2 Parameter Estimation Algorithms
   3.3 ISSf-CBF and Safety Filter
   3.4 Comparative Results of Estimators
```

```
4. Part 2: Robust Control under Actuation Uncertainty
   4.1 Robust CLF: Min-Max to QP via Duality
   4.2 Robust CBF: Min-Max to QP via Duality
   4.3 Set Membership Identification and Uncertainty Reduction
   4.4 Combined Robust Controller with SMID
5. Comprehensive Discussion
6. Conclusions
7. References (optional)
```

8 References

1. Cohen, M., & Belta, C. (2023). *Adaptive and Learning-Based Control of Safety-Critical Systems*. Springer.

2. Ames, A. D., et al. (2017). Control barrier function based quadratic programs for safety critical systems. *IEEE TAC*, 62(8), 3861–3876.

3. Chowdhary, G., et al. (2013). Exponential parameter and tracking error convergence guarantees for adaptive controllers without persistency of excitation. *IJC*, 87(8).

4. Ljung, L. (1999). *System Identification: Theory for the User*. Prentice Hall.

5. Vandenberghe, L. & Boyd, S. (1996). Semidefinite programming. *SIAM Review*, 38(1), 49–95.

6. Polyak, B. T., Nazin, S. A., Durieu, C., & Walter, E. (2004). Ellipsoidal parameter or state estimation under model uncertainty. *Automatica*, 40(7), 1171–1179.