Adaptive Control Class – Mini-Project 4

Direct Model Reference Adaptive Control, Robust Modifications, and Observer-Based Output Feedback with Adaptive Augmentation

Semester: Spring 2026 Due Date: 8 Mordad 1405

Based on Chapters 7–11, 14 of "Robust and Adaptive Control with Aerospace Applications" by E. Lavretsky and K. A. Wise, 2nd ed., Springer, 2024.

Abstract

This mini-project covers the design, analysis, and robustification of direct model reference adaptive control (MRAC) for MIMO nonlinear systems with parametric uncertainties and external disturbances. Starting from a state-feedback direct MRAC baseline, you will explore indirect adaptive schemes, augment the controller with integral action, and then apply robust modifications—dead-zone, σ/e-modification, and parameter projection—to guarantee uniform ultimate boundedness (UUB) in the presence of bounded disturbances. Finally, you will design an output-feedback servomechanism using observer-based loop transfer recovery (LTR) and adaptive augmentation. Extensive simulations under various uncertainty and disturbance scenarios are required to compare the performance and robustness of all designs.

1 Project Overview

The project is divided into six integrated tasks that systematically build a deep understanding of modern MRAC theory as presented in Lavretsky and Wise:

(1) **Direct MRAC** – state feedback with Lyapunov-based adaptation laws.

(2) **Indirect MRAC** – parameter identification plus certainty-equivalence control.

(3) **Integral Augmentation** – adding integral error states to improve steady-state tracking.

(4) **Robust Modifications** – dead-zone, σ/e-modification, and projection to prevent parameter drift and ensure UUB.

(5) **Output-Feedback Control** – observer-based loop transfer recovery (LTR) with adaptive augmentation.

(6) **Comprehensive Evaluation** – comparison under different uncertainties, disturbances, and sensor noise.

2 Learning Objectives

After completing this project, you will:

- Derive and implement direct and indirect MRAC laws for MIMO systems.
- Apply Lyapunov stability theory to prove bounded tracking and, under ideal conditions, asymptotic convergence using Barbalat's lemma.
- Design integral-augmented MRAC to reject constant disturbances.
- Employ dead-zone, σ/e-modification, and projection robust modifications to prevent parameter drift and maintain UUB in non-ideal environments.
- Synthesize an output-feedback controller using LQG/LTR with adaptive augmentation.
- Critically analyze trade-offs between performance, robustness, and computational complexity.

3 System Description and Reference Model

3.1 Plant Dynamics

Use your MIMO nonlinear system with **parametric uncertainties** and an **additive bounded disturbance** similar to:

$$\dot{x} = Ax + B\big[\Lambda u + \Theta^\top \Phi(x)\big] + \eta(x,t), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\Lambda \in \mathbb{R}^{m\times m}$ is an unknown constant nonsingular matrix, $\Theta \in \mathbb{R}^{N\times m}$ is a matrix of unknown constant parameters, $\Phi(x) \in \mathbb{R}^N$ is a known regressor vector, and $\eta(x,t)$ represents bounded external disturbances with a known bound. The pair (A, B) is assumed controllable.

3.2 Reference Model

The desired closed-loop behavior is specified by a stable linear reference model:

$$\dot{x}_m = A_m x_m + B_m r(t), \tag{2}$$

with $A_m \in \mathbb{R}^{n\times n}$ Hurwitz, $B_m \in \mathbb{R}^{n\times m}$, and $r(t)$ a bounded piecewise continuous reference command. Assume a matching condition: there exist ideal gains K_x^* and K_r^* such that

$$A + B\Lambda K_x^{*\top} = A_m, \qquad B\Lambda K_r^{*\top} = B_m. \tag{3}$$

4 Task 1: State Feedback Direct MRAC

Design a direct MRAC control law

$$u = \hat{K}_x^\top x + \hat{K}_r^\top r - \hat{\Theta}^\top \Phi(x), \tag{4}$$

where $\hat{K}_x$, $\hat{K}_r$, and $\hat{\Theta}$ are adaptive estimates. Define the tracking error $e = x - x_m$. Derive the error dynamics

$$\dot{e} = A_m e + B\Lambda\big[\tilde{K}_x^\top x + \tilde{K}_r^\top r - \tilde{\Theta}^\top \Phi(x)\big], \tag{5}$$

with $\tilde{K}_x = \hat{K}_x - K_x^*$, etc. Using the Lyapunov function candidate

$$V = e^\top P e + \mathrm{tr}\big(\tilde{K}_x^\top \Gamma_x^{-1} \tilde{K}_x \Lambda\big) + \mathrm{tr}\big(\tilde{K}_r^\top \Gamma_r^{-1} \tilde{K}_r \Lambda\big) + \mathrm{tr}\big(\tilde{\Theta}^\top \Gamma_\Theta^{-1} \tilde{\Theta}\Lambda\big), \tag{6}$$

where $P = P^\top \succ 0$ solves $A_m^\top P + P A_m = -Q$ for some $Q \succ 0$, derive the adaptive laws:

$$\dot{\hat{K}}_x = -\Gamma_x x e^\top P B, \tag{7}$$

$$\dot{\hat{K}}_r = -\Gamma_r r e^\top P B, \tag{8}$$

$$\dot{\hat{\Theta}} = \Gamma_\Theta \Phi(x) e^\top P B. \tag{9}$$

Implement these laws in MATLAB/Simulink. Show that for $\eta(x,t) = 0$, the tracking error converges to zero asymptotically (using Barbalat's lemma: $\dot{V} \le -e^\top Q e$, thus $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e}$ bounded $\implies e(t) \to 0$).

Simulate the nominal MRAC for step and sinusoidal commands, and plot $e(t)$, control effort, and parameter estimates.

Deliverable: MATLAB function `Direct_MRAC.m`, Simulink model, and a report section with derivation, Lyapunov analysis, and simulation results.

5 Task 2: Indirect MRAC

Implement an indirect adaptive controller. First, design an online parameter estimator for Λ and Θ (e.g., using the RLS or gradient algorithms from Mini-Project 3). At each time step, compute the controller gains by solving the matching equations:

$$K_x = \Lambda^{-1}(A_m - A), \qquad K_r = \Lambda^{-1} B_m. \tag{10}$$

Then apply $u = K_x^\top x + K_r^\top r - \hat{\Theta}^\top \Phi(x)$ where $\hat{\Theta}$ comes from the identification. Compare the transient behavior and parameter convergence with the direct MRAC. Discuss the effect of PE (persistency of excitation) on both schemes.

6 Task 3: Integral Augmentation

Augment the direct MRAC with integral error states to reject constant disturbances and improve steady-state tracking. Define the integral error

$$\dot{e}_I = y - y_m = Cx - C_m x_m, \tag{11}$$

and extend the state vector to $\bar{x} = [x^\top, e_I^\top]^\top$. Design an extended reference model and adaptive controller. Derive the necessary modifications to the Lyapunov analysis and adaptive laws. Simulate a step disturbance and compare the integral-augmented MRAC with the baseline.

7 Task 4: Robust Modifications

Introduce a bounded disturbance $\eta(x,t) \neq 0$ and observe parameter drift in the baseline MRAC. Implement and compare the following robust modifications to guarantee **Uniform Ultimate Boundedness (UUB)**:

7.1 Dead-Zone Modification

Modify the adaptive laws by stopping adaptation when the tracking error is small:

$$\dot{\hat{K}}_x = \begin{cases} -\Gamma_x x e^\top PB, & \text{if } \|e\| > \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

(and similarly for the other gains). Choose ε based on the disturbance bound.

7.2 σ-Modification

Add a leakage term:

$$\dot{\hat{K}}_x = -\Gamma_x x e^\top PB - \sigma \Gamma_x \hat{K}_x, \tag{13}$$

with $\sigma > 0$. Discuss how σ balances robustness and tracking performance.

7.3 e-Modification

Make the leakage term proportional to the error norm:

$$\dot{\hat{K}}_x = -\Gamma_x x e^\top PB - \sigma \|e\| \Gamma_x \hat{K}_x. \tag{14}$$

7.4 Parameter Projection

Constrain the parameter estimates to a known convex set using the projection operator:

$$\dot{\hat{K}}_x = \mathrm{Proj}_{\mathcal{K}}\big(-\Gamma_x x e^\top PB\big). \tag{15}$$

Define the admissible set and implement the projection algorithm.

Analysis: For each modification, provide a Lyapunov-based proof of UUB. Simulate the system under the same disturbance and compare:

- Tracking error norm $\|e(t)\|$.
- Control effort.
- Parameter estimation error.

Discuss the trade-offs between robustness, transient performance, and computational complexity.

8 Task 5: Output Feedback with Observer-Based LTR and Adaptive Augmentation

Assume now that only the output $y = Cx$ is available for measurement, with $C \in \mathbb{R}^{p\times n}$ $(p < n)$ and the pair (A, C) detectable. Design an **observer-based loop transfer recovery (LTR)** controller following Chapter 14 of Lavretsky and Wise.

8.1 Observer Design

Construct a Kalman filter (or an asymptotic observer) with gain L chosen such that $A - LC$ is Hurwitz and the loop transfer function recovers the robustness properties of the state-feedback design. The observer dynamics are

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{16}$$

8.2 Output Feedback MRAC Baseline

Replace the true state x in the direct MRAC law with the estimate $\hat{x}$. Analyze the stability of the combined observer–controller system.

8.3 Loop Transfer Recovery (LTR)

Tune the observer gain L (e.g., by solving an asymptotic algebraic Riccati equation) to make the open-loop transfer function at the plant input approximate that of the full-state feedback case. Demonstrate the recovery process by plotting the singular values of the loop transfer function as the recovery parameter varies.

8.4 Adaptive Augmentation

Augment the output-feedback MRAC with an additional adaptive element to improve robustness against model uncertainties and assess its impact.

9 Task 6: Comprehensive Evaluation and Discussion

Perform a systematic comparison of all controllers under the following scenarios:

- **Nominal conditions** (no disturbance, true parameters).
- **Constant disturbance** at the plant input.
- **Time-varying disturbance** (e.g., sine wave with unknown frequency).
- **Parameter step change** (simulate sudden payload drop in a robotic arm).
- **Sensor noise** (add Gaussian noise to output measurements).

For each scenario, compare the direct MRAC, indirect MRAC, integral-augmented MRAC, robust MRAC variants, and output-feedback LTR with adaptive augmentation. Quantify performance using:

- RMS tracking error $\sqrt{\frac{1}{T}\int_0^T \|e(t)\|^2 dt}$.
- Maximum control effort $\max_t \|u(t)\|$.
- Parameter estimation error $\|\Theta - \hat{\Theta}\|_F$.

Your discussion should address:

- The role of Barbalat's lemma in proving convergence of $e(t)$ in the absence of disturbances.
- How each robust modification prevents parameter drift and the associated UUB proof sketch.
- The impact of LTR recovery gain on robustness and noise sensitivity.
- Practical guidelines for selecting dead-zone thresholds, σ, and projection bounds.

10 Code Organization

All simulations must run from a single `Main.m` script. Suggested file structure:

```
Main.m                           # Master script
Plant_Dynamics.m                 # F, g, Uncertainty and disturbance model
Reference_Model.m                # Am, Bm, Cm
Direct_MRAC.m                    # Task 1: direct adaptive law
Indirect_MRAC.m                  # Task 2: online ID + certainty-equiv.
Integral_MRAC.m                  # Task 3: augmented error dynamics
Robust_Modifications/            # Task 4
   Deadzone_MRAC.m
   Sigma_MRAC.m
```

```
   eMod_MRAC.m
   Projection_Operator.m
Observer_LTR.m                 # Task 5: observer and LTR gain
OutputFeedback_MRAC.m          # Task 5: output-feedback MRAC
Adaptive_Augmentation.m        # Task 5: augmentation law
Plot_Results.m
Simulink_Models/               # All .slx files
README.txt
```

11 Report Structure

```
1. Introduction
2. System Model and Reference Model
3. Direct MRAC Design and Analysis
4. Indirect MRAC Design and Comparison
5. Integral-Augmented MRAC
6. Robust Adaptive Control Modifications
  6.1 Dead-Zone
  6.2 Sigma-Modification
  6.3 e-Modification
  6.4 Projection
7. Output Feedback with LTR and Adaptive Augmentation
8. Comprehensive Simulation Results and Discussion
9. Conclusions
10. References
```

12 Deliverables

Table 1: Project Deliverables

Deliverable	**Format**	**Details**
Handwritten Report	PDF scan	Maximum 12 pages, single-sided, legible handwriting.
Video Presentation	MP4 file	7 minutes (strict), explaining code and key results.
MATLAB/Simulink Code	`.m` and `.slx` files	All files, archived with folder structure; executable via `Main.m`.

13 References

1. Lavretsky, E., & Wise, K. A. (2024). *Robust and Adaptive Control with Aerospace Applications* (2nd ed.). Springer.

2. Slotine, J.-J. E., & Li, W. (1991). *Applied Nonlinear Control*. Prentice Hall.

3. Ioannou, P. A., & Sun, J. (2012). *Robust Adaptive Control*. Dover.

4. Khalil, H. K. (2002). *Nonlinear Systems* (3rd ed.). Prentice Hall.

5. Doyle, J. C., & Stein, G. (1981). Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE TAC*, 26(1), 4–16.