In the Name of God

Amirkabir University of Technology
(Tehran Polytechnic)
Department of Electrical Engineering

Modern Control
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Fall 2024
Homework 1
Due: Oct 8, 2024

Please notice the following:

- Write the answers to the exercises in a neat and readable manner and create a PDF file for it using the CamScanner. You can also type the answers if you prefer.

- You may use MATLAB and Simulink or Python only to solve exercises that are marked with the [**M**] symbol and the rest of the exercises must be solved through manual steps. It should be noted that there is no problem with using MATLAB to verify your answers for other exercises, but *do not use MATLAB first*. You will not gain any intuition by looking at results — you need to learn how the method works and improve your problem-solving acumen by solving the problems by hand.

- For exercises that require the use of MATLAB and Simulink, prepare a *maximum of a 3-minute* recorded video to answer each question and reduce the size of the file as much as possible. In the recorded video, describe the activities performed to obtain the solution and deliver your analysis of the results. To solve each question, it is *mandatory* to submit the written code along with the recorded video. Note that answers without a video or code will not be graded.

- Submit a compressed file with the naming format MC_HW1_FullName on the Courses platform. The file should include the answers PDF file along with any MATLAB files and videos (if applicable). Please ensure that the files are organized in their corresponding folders for each question.

- Students are expected to submit homework by 11:59 pm on the due date. However, If you are unable to submit the homework by the deadline due to any circumstances, you may still submit it up to one week late with a 20% penalty deducted from the earned grade. Submissions after one week past the due date will not be accepted. Please plan your time carefully to avoid needing this extension.

- The homework assignments are meant to be completed *individually*. While getting guidance from friends is acceptable, it is expected that you have sufficiently thought about the problem beforehand. However, any form of collaboration beyond seeking advice, such as exchanging solutions or copying code is strictly prohibited, and submitting similar answers will result in a grade of zero.

- The use of AI tools such as ChatGPT to write code is not allowed, and even if you modify the code generated by the AI, it is still detectable and will not be given any grades.

- If you have any questions regarding the exercises, please ask your questions through the Telegram group, as your question is likely a question that other friends may have as well.

Questions

1. Answer the following questions:

   a) Consider matrices $A,\ B \in\ R^{n\times n}$. Show that λ is an eigenvalue of the matrix AB if and only if it is an eigenvalue of the matrix BA.

   b) Consider matrices $A \in\ R^{n\times n}$. Assuming that the sum of the elements in each column of this matrix is equal to a real number c. Show that c is an eigenvalue of the matrix A.

   c) Square matrices $A,\ M \in\ R^{n\times n}$ are similar if there exists an invertible matrix $T \in\ R^{n\times n}$ such that:

   $$TAT^{-1} = M$$

   Show that two similar matrices have the same eigenvalues and the same characteristic equation.

2. Let A and B be arbitrary $n\times n$ matrices. Prove that if $AB = BA$, then the following inequality holds:

   $$Rank(A+B) + Rank(AB) \leq Rank(A) + Rank(B)$$

3. Consider the following matrices:

   $$A = \begin{bmatrix} 1 & 1 & 5 & 6 \\ 2 & 1 & 8 & 8 \\ 1 & 2 & 7 & 10 \\ -1 & 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

   a) Find a basis for the column space (range) and null space of the matrices A and B.

   b) Determine the rank and nullity of A.

   c) Verify that $R(B)$ is orthogonal to $N(B^T)$.

   d) Find the reduced row echelon form of the matrix A utilizing the `rref` command, and verify your answer from part b using the `null`, `orth` and `rank` commands. [**M**]

4. Find the determinant of the matrix $A_{n\times n}$ which is defined as follows:

   $$a_{i,\ j} = i + j, \qquad \text{for } n \geq 3$$

5. Consider the matrix A:

   $$A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

   a) Find the diagonal form of the matrix A.

   b) Find $\exp(At)$.

6. (Bonus) Consider the optimization problem of least squares with $\ell 2$-regularization:

   $$w^* = \text{argmin}_w f\left(w\right)$$

   $$f(w) = \frac{1}{2n}\left\|x^T w - y\right\|_2^2 + \frac{\lambda}{2}\|w\|_2^2$$

To solve the above problem, we use the gradient descent method step by step. In each step, we move in the opposite direction of the gradient to reach a local minimum for the optimization problem. The initial value is chosen randomly.

$$w^{t+1} = w^t - \alpha \nabla f(w^t)$$

The value of α is an arbitrary number and is considered as:

$$\alpha = \frac{1}{\sigma_{max}(A)}$$

where $\sigma_{max}(A)$ is the largest singular value of matrix A, which is defined as follows:

$$A = \frac{1}{n} X X^T + \lambda I$$

a) Prove that:

$$\nabla f(w) = Aw - \frac{1}{n} X y = A(w - w^*)$$

b) Prove that matrix A is positive semi-definite.

c) Prove that:

$$\left\| w^{t+1} - w^* \right\| \leq \left(1 - \frac{\sigma_{\min}(A)}{\sigma_{\max}(A)} \right) \left\| w^t - w^* \right\|$$

Good Luck
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