Mini-Project I: Linear MPC

Deadline: 30 Aban 1404

1 Project Objectives

The goal of this mini-project is to explore and compare classical LQR and model predictive control (MPC) on a chosen nonlinear multi-input multi-output (MIMO) system with constraints. The student should perform the following tasks:

1. Select a *nonlinear MIMO* plant from the literature (with a known dynamic model) that has *constraints* on inputs and/or states (attach the used reference).

2. Compute a consistent equilibrium (trim point) of the nonlinear model and linearize the model around this operating point.

3. Design an LQR controller for the linearized model.

4. Design a constrained finite-horizon MPC controller for the linearized model, including appropriate terminal cost and terminal set for stability.

5. Apply both the LQR and MPC controllers to the original nonlinear model in simulation, using multiple prediction horizons for MPC.

6. Compare the controllers in terms of closed-loop performance, optimal cost, and constraint handling.

7. Investigate the effect of terminal ingredients (terminal cost and terminal constraint set) on stability and performance.

8. Test robustness of both controllers against model uncertainty and disturbances.

The final deliverables are 1) the complete code (which should have a MAIN file that must execute ALL part of your project), 2) a written report (about

7 pages), and 3) a 7-minute presentation *video* (all these three parts are Mandatory). The report should include an introduction, description of the chosen system, design methodology, simulation results, analysis, and conclusions.

2 System Selection

Choose a nonlinear MIMO example with constraints studied in the literature. The system should have at least two inputs and multiple states, and ideally state or input constraints (hard limits) that are meaningful.

Obtain the equations of motion and parameter values from a source (paper or text). Define clearly the state vector $x(t)$, the input vector $u(t)$, and any outputs. Identify the relevant constraints (e.g. actuator saturation, state bounds) to include in the control design.

NOTE: Each student must select a *distinct* dynamic system for their project. If you wish to use a well-known model from the literature, please email me the details of your chosen model so I can confirm that no one else has selected the same one.

3 Model Linearization at Equilibrium

Find a steady-state operating point (x_e, u_e) by solving $f(x_e, u_e) = 0$ for the nonlinear model $\dot{x} = f(x, u)$. This gives the trim condition at which $\dot{x} = 0$. Let $\delta x = x - x_e$ and $\delta u = u - u_e$. Then linearize the dynamics to obtain

$$\dot{\delta x} = A_c\, \delta x + B_c\, \delta u,$$

where

$$A_c = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)}, \quad B_c = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}.$$

This linearization follows from a first-order Taylor expansion about the equilibrium. Compute A and B (analytically or via numerical differentiation). Verify that the linear pair (A_c, B_c) is controllable. Also determine the output equation $y = Cx + Du$ if needed for MPC formulation.

Then, discretize your model (using an appropriate sampling time) in order to obtain

$$\delta x(k+1) = A\, \delta x(k) + B\, \delta u(k),$$

Finally, compare the behavior of the nonlinear model, the linear continuous model, and the linear discrete model under the same set of control inputs around the equilibrium point to validate both the linearization and the discretization.

4 Terminal Cost and Constraint Design

To guarantee stability of the MPC, design suitable terminal ingredients. First compute the infinite-horizon LQR cost matrix P by solving the continuous-time Algebraic Riccati Equation

$$P = A^\top PA - (A^\top PB)(B^\top PB + R)^{-1}(B^\top PA) + Q$$

using the chosen state-weight Q and input-weight R. The matrix P is used as the terminal cost weight: adding a terminal penalty $x_N^T P x_N$ makes the finite-horizon MPC mimic the infinite-horizon LQR (ensuring stability for the unconstrained case).

To handle constraints, also define a terminal set $\mathcal{X}_f$ in which the LQR controller keeps the state admissible. For example, choose $\mathcal{X}_f = \{x : x^T P x \leq \alpha\}$ for some α, intersected with the state constraints. Impose the terminal state constraint $x_N \in \mathcal{X}_f$ in the MPC formulation. In summary, include:

- Terminal cost $V_f(x_N) = (x_N - x_e)^T P(x_N - x_e)$.
- Terminal constraint $x_N \in \mathcal{X}_f$ (a region invariant under the LQR).

Without these terminal conditions, a finite-horizon MPC can fail to stabilize the system when constraints become active.

5 LQR Controller Design

Using (A, B), select $Q \succeq 0$ and $R \succ 0$ to penalize states and inputs. Solve the LQR problem to obtain P and feedback gain $K = (B^\top PB+R)^{-1}B^\top PA$. The LQR law is

$$u = u_e - K(x - x_e).$$

Verify that the closed-loop matrix $(A - BK)$ has stable eigenvalues. LQR will stabilize the linear model (and locally the nonlinear one) if $Q, R > 0$. Simulate the linearized system under this LQR control to get baseline performance metrics (settling time, overshoot, control effort).

6 MPC Controller Design

Formulate a discrete-time MPC for the linear model. The finite-horizon cost is

$$J = \sum_{k=0}^{N-1} \Big[(x_k - x_e)^T Q (x_k - x_e) + (u_k - u_e)^T R (u_k - u_e)\Big] + (x_N - x_e)^T P (x_N - x_e).$$

Enforce the dynamics $\delta x_{k+1} = A\delta x_k + B\delta u_k$ and the constraints $u_k \in \mathcal{U}$, $x_k \in \mathcal{X}$ at each step, plus $x_N \in \mathcal{X}_f$. Use the same Q, R as LQR and terminal weight P as above. Implement the optimization in software (e.g. YALMIP/CVX, etc.).

For a linear system without constraints, a sufficiently long-horizon MPC with terminal cost will recover LQR performance. Try horizons $N = 5, 10, \ldots$ and check how solution feasibility and performance change. In general, longer horizons improve performance and enlarge the domain of attraction (feasible set), at the cost of higher computation.

7 Nonlinear Simulation and Comparison

Simulate the original nonlinear plant with the designed controllers. For LQR, use $u = u_e - K(x - x_e)$ at each step. For MPC, at each sampling instant solve the optimization and apply the first input. Compare for each controller and horizon:

- State and input trajectories (plot $x(t)$ and $u(t)$).
- Constraint satisfaction (did any controller violate bounds?).
- Regulation performance (settling time, overshoot).
- Accumulated cost (sum of $x^T Q x + u^T R u$ over time).

Repeat for multiple initial conditions if possible.

8 Effect of Terminal Ingredients

Test how terminal cost and constraints affect MPC. For example, run MPC with:

- No terminal cost ($P = 0$) and no terminal constraint.

- Terminal cost included but no terminal set constraint.
- Both terminal cost and terminal constraint enforced.

Run simulations and observe stability/performance. Without terminal conditions, a short-horizon MPC may become unstable or suboptimal (it may not converge to equilibrium). Show that including the LQR-based terminal cost and set stabilizes the controller and allows shorter horizons to work.

9 Robustness Testing

Evaluate robustness by introducing model uncertainties or disturbances. For instance, perturb a plant parameter by 10% or add an external disturbance to the nonlinear model. Apply the same LQR and MPC controllers (designed for the nominal model) to this perturbed plant. Compare how each controller handles the mismatch: observe any loss of performance or constraint violations. Robustness is critical; "robustness to inaccurate model predictions is usually more important than nominal performance".

10 Report and Presentation

Prepare a 7-page report describing your work. Include:

- The chosen nonlinear system model (with reference).
- Computation of the trim point and the linearized (A, B).
- LQR design (chosen Q, R, feedback gain K) and performance results.
- MPC design (horizon N, weights Q, R, terminal P, constraints).
- The cost function and any terminal region definitions.
- Simulation results: plots comparing LQR vs MPC for different N and designs.
- Discussion of stability, effect of terminal ingredients, and robustness observations.
- Conclusions summarizing key findings.

Also prepare a 7-minute video summarizing the methodology and key results (screen capture of plots or slides is recommended).

Cite relevant sources for the system model and control theory. Ensure clarity and reproducibility of your results.