Mini-Project 4: Robust Model Predictive Control

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1 Introduction and Objectives

This fourth mini-project marks the culmination of your study of Model Predictive Control by venturing into the critical area of **Robust MPC (RMPC)**. In real-world applications, models are never perfect, and systems are always subject to external disturbances. The nominal MPC designed in previous projects guarantees stability and performance only for the idealized, disturbance-free model. When uncertainty is present, its performance can degrade, and worse, its guarantees of constraint satisfaction can be violated.

The primary objective of this project is to design, implement, and critically evaluate different RMPC strategies for the same nonlinear plant and trajectory-following task from your previous projects (**with the same quadratic trajectory as that of your previous project**). You will explore both open-loop and closed-loop robust strategies and compare their effectiveness, computational complexity, and the size of their safe operating regions.

Learning Goals:

- Understand the fundamental principles of ensuring robustness in MPC.
- Formulate and solve min-max optimization problems for open-loop RMPC.
- Implement a state-of-the-art Tube-based MPC, a closed-loop robust strategy.
- Develop a methodology for estimating and comparing the Region of Attraction (ROA) for different controllers.
- Systematically compare advanced controllers against simpler baselines (Nominal MPC, LQR) using meaningful performance metrics.

2 System and Problem Formulation

2.1 System Model with Uncertainty

You will continue to use the nonlinear discrete-time system from your previous projects. However, we now introduce an additive, bounded process disturbance w_k. The system dynamics are given by:

$$x_{k+1} = f(x_k, u_k) + Dw_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_w}$ is an unknown but bounded disturbance.

Uncertainty Set: The disturbance w_k is assumed to belong to a known compact, polytopic set $\mathbb{W} \subset \mathbb{R}^{n_x}$ for all $k \geq 0$. This set is typically defined by a set of linear inequalities:

$$\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid Vw \leq \mathbf{1}\} \tag{2}$$

You are required to define a suitable polytope $\mathbb{W}$ for your system. For example, a simple box constraint set is a valid polytope.

2.2 Control Objective and Constraints

The control objective remains the same: track the quadratic reference trajectory provided in previous assignments. The system is subject to the same state and input constraints:

$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U} \tag{3}$$

A key challenge of RMPC is to satisfy these constraints for *all possible realizations* of the disturbance sequence.

3 Part 1: Open-Loop Optimization (Min-Max RMPC)

In this part, you will design RMPC controllers based on an open-loop prediction philosophy. At each time step k, the controller computes a fixed sequence of future control inputs $\mathbf{u}_k = \{u_{0|k}, u_{1|k}, \ldots, u_{N-1|k}\}$ that is robust to all possible disturbance sequences.

3.1 Theoretical Background

You should design and apply the RMPC based on chapter 3 of the book, where the model of the system is obtained using **successive linearization** at each time step around the current operating point.

3.2 Task 1.1: RMPC with Nominal Cost

Design and implement an RMPC that minimizes the nominal cost function, while ensuring robust constraint satisfaction.

3.3 Task 1.2: RMPC with Worst-Case Cost

Now, design a more conservative RMPC that minimizes the *worst-case* cost over all possible disturbance sequences. The problem becomes a min-max problem.

3.4 Task 1.3: Analysis of Open-Loop RMPC

For both controllers designed above:

1. Implement the controllers in your simulation environment.

2. Numerically estimate the Region of Attraction (ROA) for each controller. To do this, create a grid of initial states $x(0)$ around the reference starting point. The ROA is the set of initial states from which the RMPC optimization problem is feasible.

3. Repeat the ROA estimation for different prediction horizons (e.g., $N \in \{5, 10, 15\}$).
4. Plot and compare the ROAs. Discuss how the choice of cost function (nominal vs. worst-case) and the prediction horizon N affect the size of the ROA and the controller's conservatism.

4 Part 2: Closed-Loop Optimization (Parameterized Tube MPC)

Open-loop strategies can be overly conservative because they do not account for future measurements. Tube-based MPC is a closed-loop strategy that defines a "tube" around a nominal trajectory, within which the real state is guaranteed to remain. We will implement the Parameterized Tube MPC as described in Section 4.2 of Kouvaritakis & Cannon, "Model Predictive Control".

4.1 Task 2.1: Design of Parameterized Tube MPC

The optimization problem for the Tube MPC is according to Eq. (4.49) of the book. Again you should used the linearized model which is obtained at each step around the current operating point of the system.

4.2 Task 2.2: Performance and ROA Analysis

1. Implement the Parameterized Tube MPC controller.
2. Numerically estimate its Region of Attraction (ROA) using the same grid-based method as in Part 1.
3. Directly compare the size of the Tube MPC's ROA against the ROAs of the open-loop controllers from Part 1.
4. Run a simulation of the trajectory following task. Compare the closed-loop performance (tracking error, cost) against the open-loop RMPC methods. Is Tube MPC less conservative? Justify your answer with plots and data.

5 Part 3: Comparative Study and Final Analysis

In this final part, you will perform a comprehensive comparison of all the controllers you have designed against standard benchmarks.

5.1 Controllers for Comparison

You should have the following controllers ready for simulation:

1. **LQR Controller:** A baseline linear controller designed by linearizing the system at the final target setpoint of the trajectory.
2. **Nominal MPC:** Your MPC from previous projects, which ignores disturbances.

3. **Open-Loop RMPC:** The better performing of your two designs from Part 1.
4. **Tube-Based RMPC:** The controller from Part 2.

5.2 Simulation Scenarios

Run all four controllers on the trajectory following task under the following conditions:

1. **Stochastic Disturbances:** At each step, draw a random disturbance vector w_k from within your defined polytope $\mathbb{W}$ (e.g., using a uniform distribution).
2. **Worst-Case Disturbances:** If possible, manually craft a disturbance sequence that is adversarial (i.e., tries to push the state towards the constraint boundary).
3. **Unmodelled Dynamics (Optional):** Test the controllers with a disturbance that is slightly outside the defined set $\mathbb{W}$ to see how they fail.

5.3 Performance Metrics

For your report and video, you must compare the controllers using a clear set of metrics. Your analysis should be quantitative. Suggested metrics include:

- **Mean Squared Tracking Error (MSTE):** $\frac{1}{T}\sum_{k=0}^{T-1}\|x_k - r_k\|_2^2$.
- **Total Cost:** The accumulated stage cost over the simulation horizon.
- **Control Effort:** $\frac{1}{T}\sum_{k=0}^{T-1}\|u_k\|_2^2$.
- **Constraint Violation:** Report if and by how much any controller violates the state/input constraints.
- **Computational Time:** Measure the average time required to solve the MPC optimization at each step.
- **Region of Attraction (ROA):** The final comparison of the ROA plots.

6 Deliverables

Your project submission must consist of three parts:

1. **Source Code (ZIP file):**
   - A single main script (e.g., `main.m` or `main.py`) that, when run, executes all necessary simulations and generates all figures used in your report.
   - All other functions, scripts, and class files required to run the main script.
   - The code must be clean, well-commented, and easy to understand.
2. **Project Report (PDF, max 10 pages):**
   - A concise but complete document detailing your work.

- **Structure:** Introduction, Problem Formulation, Part 1 Methodology and Results, Part 2 Methodology and Results, Part 3 Comparative Analysis, and Conclusion.
- The report must explain your design choices, present your results clearly using plots and tables, and provide insightful discussion and analysis.

3. **Video Presentation (MP4, max 10 minutes):**
   - A screen-capture video where you verbally explain your project.
   - The video should include a live run of your main simulation script, showcasing how the results are generated.
   - Briefly explain the theory behind the controllers, walk through the key parts of your code, and summarize the results and conclusions from your report.